Representation functions for Jordanian quantum group $SL_h(2)$ and Jacobi polynomials

2000 J. Phys. A: Math. Gen. 33 3735

(http://iopscience.iop.org/0305-4470/33/20/302)

J. Phys. A: Math. Gen. **33** (2000) 3735–3752. Printed in the UK PII: S0305-4470(00)11136-9

Representation functions for Jordanian quantum group $SL_h(2)$ and Jacobi polynomials

N Aizawa†

Department of Applied Mathematics, Osaka Women's University, Sakai, Osaka 590-0035, Japan

Received 18 January 2000

Abstract. The explicit expressions of the representation functions (D-functions) for Jordanian quantum group $SL_h(2)$ are obtained by combination of tensor operator technique and Drinfeld twist. It is shown that the D-functions can be expressed in terms of Jacobi polynomials as the undeformed D-functions can. Some of the important properties of the D-functions for $SL_h(2)$ such as Wigner's product law, recurrence relations and RTT-type relations are also presented.

1. Introduction

It is known that quantum deformation of Lie group $GL(2)$ with central quantum determinant is classified into two types [1]: the standard deformation $GL_q(2)$ [2] and the Jordanian deformation $GL_h(2)$ [3–5]. The representation theory of $GL_q(2)$ has been studied extensively and we know that its contents are quite rich (see, for instance, [6, 7]). On the other hand, the representation theory of $GL_h(2)$ has not been developed yet. There are some works studying differential geometry on the quantum h-plane and on $SL_h(2)$ itself [8]. However, the representation functions for $GL_h(2)$, the most basic ingredient of representation theories, have not been known. Recently, Chakrabarti and Quesne [9] showed that the representation functions for two-parametric extension of $GL_h(2)$ [5, 10] can be obtained from the standard deformed ones via a contraction method and gave the explicit form of the representation functions for some low-dimensional cases. In [11], the present author shows that the Jordanian deformation of symplecton for $sl(2)$ gives a natural basis for a representation of $SL_h(2)$ and he also gives another basis in terms of the quantum h-plane.

The purpose of this paper is to obtain explicit formulae for $SL_h(2)$ representation functions using the tensor operator technique and to investigate their properties. Representation functions are also called Wigner D-functions in physicist's terminology. We use both terms and restrict ourselves to the finite-dimensional highest-weight irreducible representations of $SL_h(2)$ in this paper. In order to make a comparison between D-functions for $SL_q(2)$ and $SL_h(2)$, let us recall some known properties of D-functions for $SL_q(2)$‡: (a) Wigner's product law [13], (b) recurrence relations [13, 14], (c) orthogonality (d) RTT-type relations [14], (e) the fact that D-functions can be written in terms of the little q-Jacobi polynomials [15] and (f) the generating function [16]. We will show that many of these have counterparts in the representation theory of $SL_h(2)$. The only exception is the generating function, which is

† Present address: Arnold Sommerfeld Institute for Mathematical Physics, TU Clausthal, Leibnizstr. 10, D-38678 Clausthal-Zellerfeld, Germany. E-mail: ptaizawa@pt.tu-clausthal.de.

‡ For the D-functions for $SL_q(2)$, see [12].

not presented in this paper. Of course, this does not mean that the generating function for the D-functions of $SL_h(2)$ does not exist.

The plan of this paper is as follows: we present the definitions of $SL_h(2)$ and its dual quantum algebra $\mathcal{U}_h(sl(2))$ in section 2. In section 3, before deriving the explicit formulae for the representation functions, we discuss general features of them which are valid for any kind of deformation of $SL(2)$ under the assumption that the representation theory of the dual quantum algebra has a one-to-one correspondence with the undeformed $sl(2)$. Then we give the recurrence relations for $SL_h(2)$ D-functions. Section 4 briefly reviews the D-functions for Lie group $SL(2)$ (and $GL(2)$). We emphasize that the D-functions for $GL(2)$ form, in a certain boson realization, irreducible tensor operators of the Lie algebra $gl(2) \oplus gl(2)$. In section 5, a tensor operator technique is used to obtain the boson realization of the generators of the Jordanian quantum group $GL_h(2)$; it is then generalized to obtain the D-functions for $GL_h(2)$. We shall apply the same technique to show that the D-functions for $SL_h(2)$ can be expressed in terms of Jacobi polynomials. This method will be applied to obtain a boson realization for two-parametric extension of the Jordanian deformation of $GL(2)$ in section 6. Section 7 contains concluding remarks.

2. $SL_h(2)$ and its dual

The Jordanian quantum group $GL_h(2)$ is generated by four elements x, y, u and v subject to the relations [3–5]

$$\begin{aligned} &[v, x] = hv^2 \qquad [u, x] = h(D - x^2) \\ &[v, y] = hv^2 \qquad [u, y] = h(D - y^2) \\ &[x, y] = h(xv - yv) \qquad [v, u] = h(xv + vy) \end{aligned} \tag{2.1}$$

where $D = xy - uv - hxv$ is the quantum determinant generating the centre of $GL_h(2)$. This is a Hopf algebra and Hopf algebra mappings have a similar form as $GL_q(2)$. However, explicit form of the mappings is not necessary in the following discussion. By setting $D = 1$, we obtain $SL_h(2)$ from $GL_h(2)$.

The quantum algebra dual to $GL_h(2)$ is denoted by $\mathcal{U}_h(gl(2))$, and defined by the same commutation relations as the Lie algebra $gl(2)$

$$[J_0, J_\pm] = \pm 2J_\pm \qquad [J_+, J_-] = J_0 \qquad [Z, \bullet] = 0. \tag{2.2}$$

However, their Hopf algebra mappings are modified via twisting [17] by the invertible element $\mathcal{F} \in \mathcal{U}_h(gl(2))^{\otimes 2}$ [18]

$$\mathcal{F} = \exp(-\tfrac{1}{2} J_0 \otimes \sigma) \qquad \sigma = -\ln(1 - 2hJ_+). \tag{2.3}$$

The coproduct Δ, co-unit ϵ and antipode S for $\mathcal{U}_h(gl(2))$ are obtained from those for $gl(2)$ by

$$\Delta = \mathcal{F}\Delta_0\mathcal{F}^{-1} \qquad \epsilon = \epsilon_0 \qquad S = \mu S_0 \mu^{-1} \tag{2.4}$$

where the mappings with subscript 0 stand for the Hopf algebra mappings for $gl(2)$. The elements μ and μ^{-1} are defined, using the product m for $gl(2)$, by

$$\mu = m(\mathrm{id} \otimes S_0)(\mathcal{F}) \qquad \mu^{-1} = m(S_0 \otimes \mathrm{id})(\mathcal{F}^{-1}). \tag{2.5}$$

The twist element $\mathcal{F}$ is not dependent on the central element Z so that the Hopf algebra mappings for Z remain undeformed. Therefore, the Jordanian quantum algebra obtained by the twist element (2.3) has the decomposition $\mathcal{U}_h(gl(2)) = \mathcal{U}_h(sl(2)) \oplus u(1)$. The Jordanian quantum algebra $\mathcal{U}_h(gl(2))$ is a triangular Hopf algebra whose universal R-matrix is given by $\mathcal{R} = \mathcal{F}_{12}\mathcal{F}^{-1}$.

It is obvious, from the commutation relation (2.2), that $\mathcal{U}_h(gl(2))$ and $gl(2)$ have the same finite-dimensional highest-weight irreducible representations. Furthermore, we can easily see that tensor product of two irreducible representations (irreps) is completely reducible and decomposed into irreps in the same way as $gl(2)$, since the Clebsch–Gordan coefficients (CGCs) for $\mathcal{U}_h(gl(2))$ are the product of the ones for $gl(2)$ and the matrix elements of the twist element $\mathcal{F}$. For the $\mathcal{U}_h(sl(2))$ sector, this is carried out in [19]. The CGCs for $\mathcal{U}_h(sl(2))$ in another basis are discussed in [20].

Let Δ, ϵ be the coproduct and co-unit for $GL_h(2)$, respectively. We use the same notation for the Hopf algebra mappings of both $GL_h(2)$ and $\mathcal{U}_h(gl(2))$; however, this should not cause serious confusion. A vector space (representation space) V is called the right $GL_h(2)$ comodule, if there exists a map $\rho : V \to V \otimes GL_h(2)$ such that the following relations are satisfied:

$$(\rho \otimes \mathrm{id}) \circ \rho = (\mathrm{id}_V \otimes \Delta) \circ \rho \qquad (\mathrm{id}_V \otimes \epsilon) \circ \rho = \mathrm{id}_V \tag{2.6}$$

where id_V stands for the identity map in V. The left comodule is defined in a similar manner. Using the bases $\{e_i \| i = 1, 2, \ldots, n\}$ of V, the map ρ is written as

$$\rho(e_i) = \sum_j e_j \otimes \mathcal{D}_{ji}. \tag{2.7}$$

It follows that the relation (2.6) are rewritten as

$$\Delta(\mathcal{D}_{ij}) = \sum_k \mathcal{D}_{ik} \otimes \mathcal{D}_{kj} \qquad \epsilon(\mathcal{D}_{ij}) = \delta_{ij}. \tag{2.8}$$

We call $\mathcal{D}_{ij} \in GL_h(2)$ satisfying (2.7) and (2.8) the D-function for $GL_h(2)$.

3. Properties of D-functions

3.1. Wigner's product law and RTT-type relations

Before deriving the explicit formulae for $SL_h(2)$ D-functions, we discuss some important properties of D-functions such as Wigner's product law, recurrence relations, RTT-type relations and so on, using the definition of the universal T-matrix [21, 22]. The explicit expression of the universal T-matrix is not necessary. The universal T-matrix for the standard deformation of $GL(2)$ is given in [22], while it is not known for the Jordanian deformation of $GL(2)$.

The discussion in this section is quite general. We present it so as to be applicable to any kind of deformation of $SL(2)$ (standard, Jordanian, two-parametric extension, anything else (if any)). Then, in section 3.2 we give the results explicitly for the Jordanian deformation of $SL(2)$. It will also be seen that the discussion is easily extended to other groups.

Let $\mathcal{G}$ and $\boldsymbol{g}$ be deformation of Lie group $SL(2)$ and Lie algebra $sl(2)$, respectively. The duality between $\mathcal{G}$ and $\boldsymbol{g}$ are expressed, by choosing suitable bases, in terms of the universal T-matrix [22]. Let x^α and X_α be elements of a basis of $\mathcal{G}$ and $\boldsymbol{g}$, respectively. They are chosen as follows: the product is given by

$$x^\alpha x^\beta = \sum_\gamma h_\gamma^{\alpha,\beta} x^\gamma \qquad X_\alpha X_\beta = \sum_\gamma f_{\alpha,\beta}^\gamma X_\gamma \tag{3.1}$$

and the coproduct is given by

$$\Delta(x^\alpha) = \sum_{\beta,\gamma} f_{\beta,\gamma}^\alpha x^\beta \otimes x^\gamma \qquad \Delta(X_\alpha) = \sum_{\beta,\gamma} h_\alpha^{\beta,\gamma} X_\beta \otimes X_\gamma. \tag{3.2}$$

Then the universal T-matrix $\mathcal{T}$ is defined by

$$\mathcal{T} = \sum_\alpha x^\alpha \otimes X_\alpha. \tag{3.3}$$

We assume that the deformed algebra $\boldsymbol{g}$ has the same finite-dimensional highest-weight irreps as $sl(2)$: that is (1) each irrep is classified by the spin j and a irrep basis $|jm\rangle$ is specified by j and the magnetic quantum number m, and (2) tensor product of irreps j_1 and j_2 is completely reducible:

$$j_1 \otimes j_2 = j_1 + j_2 \oplus j_1 + j_2 - 1 \oplus \cdots \oplus |j_1 - j_2|.$$

We further assume that vectors $|jm\rangle$ are complete and orthonormal. Then the D-functions for $\mathcal{G}$ are obtained by

$$\mathcal{D}^j_{m',m} = \langle jm'|\mathcal{T}|jm\rangle = \sum_\alpha x^\alpha \langle jm'|X_\alpha|jm\rangle. \tag{3.4}$$

For the standard two-parametric deformation of $GL(2)$, the RHS of (3.4) was computed and it was shown that (3.4) coincided with the D-functions obtained by another method [23]. In our case, we show that the D-functions (3.4) satisfy (2.8) by making use of relations (3.1) and (3.2). The coproduct of $\mathcal{D}^j_{m',m}$ is computed as

$$\begin{aligned}\Delta(\mathcal{D}^j_{m',m}) &= \sum_\alpha \Delta(x^\alpha)\langle jm'|X_\alpha|jm\rangle = \sum_{\beta,\gamma} x^\beta \otimes x^\gamma \langle jm'|X_\beta X_\gamma|jm\rangle \\ &= \sum_{\beta,\gamma,k} x^\beta \otimes x^\gamma \langle jm'|X_\beta|jk\rangle\langle jk|X_\gamma|jm\rangle = \sum_k \mathcal{D}^j_{m',k} \otimes \mathcal{D}^j_{k,m}.\end{aligned}$$

To compute the co-unit for $\mathcal{D}^j_{m',m}$, we use the identity obtained from the definition of co-unit:

$$\sum_{\beta,\gamma} f^\alpha_{\beta,\gamma}\epsilon(x^\beta)x^\gamma = x^\alpha. \tag{3.5}$$

Using this relation, the universal T-matrix is rewritten as

$$\begin{aligned}\mathcal{T} &= \sum_\alpha x^\alpha \otimes X_\alpha = \sum f^\alpha_{\beta,\gamma}\epsilon(x^\beta)x^\gamma \otimes X_\alpha = \sum \epsilon(x^\beta)x^\gamma \otimes X_\beta X_\gamma \\ &= \left(\sum_\beta \epsilon(x^\beta) \otimes X_\beta\right)\mathcal{T}.\end{aligned}$$

It follows that

$$\left(\sum_\beta \epsilon(x^\beta) \otimes X_\beta\right) = (\epsilon \otimes \mathrm{id})(\mathcal{T}) = 1. \tag{3.6}$$

Therefore, the co-unit for D-functions is

$$\epsilon(\mathcal{D}^j_{m',m}) = \langle jm'|(\epsilon \otimes \mathrm{id})(\mathcal{T})|jm\rangle = \langle jm'|jm\rangle = \delta_{m',m}.$$

We first show that the D-functions (3.4) satisfy analogous relations to Wigner's product law. Let us denote the CGC for $\boldsymbol{g}$ by $\Omega^{j_1,j_2,j}_{m_1,m_2,m}$, i.e.

$$|(j_1 j_2)jm\rangle = \sum_{m_1,m_2} \Omega^{j_1,j_2,j}_{m_1,m_2,m}|j_1 m_1\rangle \otimes |j_2 m_2\rangle. \tag{3.7}$$

We write the inverse of the above relation as follows:

$$|j_1 m_1\rangle \otimes |j_2 m_2\rangle = \sum_{j,m} \mho^{j_1,j_2,j}_{m_1,m_2,m}|(j_1 j_2)jm\rangle. \tag{3.8}$$

Then the following theorem is an analogue of Wigner's product law.

Theorem 3.1. *The D-functions for $\mathcal{G}$ satisfy the relation*

$$\delta_{j,j'}\mathcal{D}^j_{m',m} = \sum_{k_1,k_2,m_1,m_2} \mho^{j_1,j_2,j'}_{k_1,k_2,m'}\Omega^{j_1,j_2,j}_{m_1,m_2,m}\mathcal{D}^{j_1}_{k_1,m_1}\mathcal{D}^{j_2}_{k_2,m_2}. \tag{3.9}$$

Proof. Because of (3.1) and (3.2), one can show that

$$(\mathrm{id} \otimes \Delta)(\mathcal{T}) = \sum_{\alpha,\beta} x^\alpha x^\beta \otimes X_\alpha \otimes X_\beta \qquad (\Delta \otimes \mathrm{id})(\mathcal{T}) = \sum_{\alpha,\beta} x^\alpha \otimes x^\beta \otimes X_\alpha X_\beta. \tag{3.10}$$

It follows that

$$(\mathrm{id} \otimes \Delta)(\mathcal{T})|(j_1 j_2) j m\rangle = \sum_{\alpha,\beta,m_1,m_2} \Omega^{j_1,j_2,j}_{m_1,m_2,m} x^\alpha x^\beta \otimes X_\alpha |j_1 m_1\rangle \otimes X_\beta |j_2 m_2\rangle. \tag{3.11}$$

The LHS of (3.11) is rewritten as

$$\sum_{m'} \mathcal{D}^j_{m',m} \otimes |(j_1 j_2) j m'\rangle = \sum_{m',k_1,k_2} \Omega^{j_1,j_2,j}_{k_1,k_2,m'} \mathcal{D}^j_{m',m} \otimes |j_1 k_1\rangle \otimes |j_2 k_2\rangle.$$

The RHS of (3.11) is rewritten as

$$\sum \Omega^{j_1,j_2,j}_{m_1,m_2,m} \langle j_1 k_1 | X_\alpha | j_1 m_1\rangle \langle j_2 k_2 | X_\beta | j_2 m_2\rangle x^\alpha x^\beta \otimes |j_1 k_1\rangle \otimes |j_2 k_2\rangle$$
$$= \sum \Omega^{j_1,j_2,j}_{m_1,m_2,m} \mathcal{D}^{j_1}_{k_1,m_1} \mathcal{D}^{j_2}_{k_2,m_2} \otimes |j_1 k_1\rangle \otimes |j_2 k_2\rangle.$$

Thus we obtain

$$\sum_{m'} \Omega^{j_1,j_2,j}_{k_1,k_2,m'} \mathcal{D}^j_{m',m} = \sum_{m_1,m_2} \Omega^{j_1,j_2,j}_{m_1,m_2,m} \mathcal{D}^{j_1}_{k_1,m_1} \mathcal{D}^{j_2}_{k_2,m_2}. \tag{3.12}$$

Using the orthogonality of $\Omega^{j_1,j_2,j}_{m_1,m_2,m}$ and $\mho^{j_1,j_2,j}_{m_1,m_2,m}$, the theorem is proved. □

Corollary 3.2. *The D-functions also satisfy the following relations:*

$$\sum_{m'} \Omega^{j_1,j_2,j}_{k_1,k_2,m'} \mathcal{D}^j_{m',m} = \sum_{m_1,m_2} \Omega^{j_1,j_2,j}_{m_1,m_2,m} \mathcal{D}^{j_1}_{k_1,m_1} \mathcal{D}^{j_2}_{k_2,m_2} \tag{3.13}$$

$$\sum_{m} \mho^{j_1,j_2,j}_{m_1,m_2,m} \mathcal{D}^j_{m',m} = \sum_{k_1,k_2} \mho^{j_1,j_2,j}_{k_1,k_2,m'} \mathcal{D}^{j_1}_{k_1,m_1} \mathcal{D}^{j_2}_{k_2,m_2} \tag{3.14}$$

$$\mathcal{D}^{j_1}_{k_1,m_1} \mathcal{D}^{j_2}_{k_2,m_2} = \sum_{j,m,m'} \mho^{j_1,j_2,j}_{m_1,m_2,m} \Omega^{j_1,j_2,j}_{k_1,k_2,m'} \mathcal{D}^j_{m',m}. \tag{3.15}$$

Proof. Equation (3.13) has already been obtained in the proof of theorem 3.1, see (3.12). The others can be obtained from (3.13) by the orthogonality of Ω and $\mho$. □

For $\mathcal{G} = SL_q(2)$ and $\boldsymbol{g} = \mathcal{U}_q(sl(2))$, the CGCs $\Omega, \mho$ are given by the q-analogue of the CGC of $sl(2)$: $\Omega^{j_1,j_2,j}_{m_1,m_2,m} = \mho^{j_1,j_2,j}_{m_1,m_2,m} = {}_qC^{j_1,j_2,j}_{m_1,m_2,m}$. From (3.9) and (3.13)–(3.15), the recurrence relations and the orthogonality of $SL_q(2)$ D-functions are obtained [6, 13, 14].

Next we show that the D-functions (3.4) satisfy the RTT-type relation.

Theorem 3.3. *The D-functions for $\mathcal{G}$ satisfy*

$$\sum_{s_1,s_2} (R^{j_1,j_2})^{s_1,s_2}_{m_1,m_2} \mathcal{D}^{j_1}_{s_1,k_1} \mathcal{D}^{j_2}_{s_2,k_2} = \sum_{s_1,s_2} \mathcal{D}^{j_2}_{m_2,s_2} \mathcal{D}^{j_1}_{m_1,s_1} (R^{j_1,j_2})^{k_1,k_2}_{s_1,s_2} \tag{3.16}$$

where $(R^{j_1,j_2})^{s_1,s_2}_{m_1,m_2}$ are the matrix elements of the universal R-matrix for $\boldsymbol{g}$:

$$(R^{j_1,j_2})^{s_1,s_2}_{m_1,m_2} = \langle j_1 m_1| \otimes \langle j_2 m_2 | \mathcal{R} | j_1 s_1\rangle \otimes |j_2 s_2\rangle.$$

Remark. For $j_1 = j_2 = \frac{1}{2}$, the matrix elements for $\mathcal{R}$ are evaluated in the fundamental representation of $\boldsymbol{g}$. Therefore, (3.16) is reduced to the defining relation of $\mathcal{G}$ in FRT formalism [2]. This implies that $\mathcal{D}^{\frac{1}{2}}_{m',m}$ are generators of $\mathcal{G}$.

Proof. The relation (3.16) can be proved by evaluating matrix elements of the RTT-type relation for the universal T-matrix [24]. We define

$$\mathcal{T}_1 = \sum x^\alpha \otimes X_\alpha \otimes 1 \qquad \mathcal{T}_2 = \sum x^\alpha \otimes 1 \otimes X_\alpha.$$

Then

$$\mathcal{T}_1\mathcal{T}_2 = \sum_{\alpha,\beta} x^\alpha x^\beta \otimes X_\alpha \otimes X_\beta = \sum_\alpha x^\alpha \otimes \Delta(X_\alpha)$$

$$\mathcal{T}_2\mathcal{T}_1 = \sum_{\alpha,\beta} x^\beta x^\alpha \otimes X_\alpha \otimes X_\beta = \sum_\alpha x^\alpha \otimes \Delta'(X_\alpha)$$

where Δ' stands for the opposite coproduct. It follows that

$$\mathcal{T}_2\mathcal{T}_1 = \sum_\alpha x^\alpha \otimes \mathcal{R}\Delta(X_\alpha)\mathcal{R}^{-1}.$$

Thus we obtain

$$(1 \otimes \mathcal{R})\mathcal{T}_1\mathcal{T}_2 = \mathcal{T}_2\mathcal{T}_1(1 \otimes \mathcal{R}).$$

Evaluating the matrix elements on $1 \otimes |j_1k_1\rangle \otimes |j_2k_2\rangle$, the theorem is proved. □

For $\mathcal{G} = SL_q(2)$, the relaiton (3.16) was proved by Nomura [14]. However, theorem 3.3 shows that (3.16) holds for any kind of deformation of $SL(2)$. In [14], the D-functions for $SL_q(2)$ are interpreted as the wavefunctions of quantum symmetric tops in noncommutative space.

3.2. Recurrence relations and orthogonality-like relations

In this section, the recurrence relations and the orthogonality-like relations of the $SL_h(2)$ D-functions are derived as a consequence of the theorems in the previous section. It is known that the CGCs for $\mathcal{U}_h(sl(2))$ are given in terms of the CGCs for $sl(2)$ and the matrix elements of the twist element $\mathcal{F}$

$$\Omega^{j_1,j_2,j}_{m_1,m_2,m} = \sum_{s_1,s_2} C^{j_1,j_2,j}_{s_1,s_2,m}(F^{j_1,j_2})^{s_1,s_2}_{m_1,m_2} \tag{3.17}$$

where $C^{j_1,j_2,j}_{s_1,s_2,m}$ is the CGC for $sl(2)$ and $(F^{j_1,j_2})^{s_1,s_2}_{m_1,m_2}$ is given by

$$(F^{j_1,j_2})^{s_1,s_2}_{m_1,m_2} = \langle j_1, m_1| \otimes \langle j_2, m_2|\mathcal{F}|j_1, s_1\rangle \otimes |j_2, s_2\rangle.$$

The explicit formula for $(F^{j_1,j_2})^{s_1,s_2}_{m_1,m_2}$ and the next relation are found in [11]:

$$(F^{j_1,j_2})^{-s_1,-s_2}_{-m_1,-m_2} = ((F^{-1})^{j_1,j_2})^{m_1,m_2}_{s_1,s_2}. \tag{3.18}$$

The CGCs for $\mathcal{U}_h(sl(2))$ satisfy the orthogonality relations [20] because of

$$\mho^{j_1,j_2,j}_{m_1,m_2,m} = (-1)^{j_1+j_2-j}\Omega^{j_1,j_2,j}_{-m_1,-m_2,-m} = \sum_{s_1,s_2} C^{j_1,j_2,j}_{s_1,s_2,m}((F^{-1})^{j_1,j_2})^{m_1,m_2}_{s_1,s_2}. \tag{3.19}$$

The relation (3.18) and the well known property of the $sl(2)$ CGC are used in the last equality.

Note that we have known the following fact because of the remark to theorem 3.3.

Proposition 3.4. *$\mathcal{D}^{\frac{1}{2}}_{m',m}$ are the generators of $SL_h(2)$*

$$\begin{pmatrix} \mathcal{D}^{\frac12}_{\frac12,\frac12} & \mathcal{D}^{\frac12}_{\frac12,-\frac12} \\ \mathcal{D}^{\frac12}_{-\frac12,\frac12} & \mathcal{D}^{\frac12}_{-\frac12,-\frac12} \end{pmatrix} = \begin{pmatrix} x & u \\ v & y \end{pmatrix}. \tag{3.20}$$

Let us consider the case that j_1 is arbitrary and $j_2 = \frac{1}{2}$ in order to derive the recurrence relations for $SL_h(2)$ D-functions. In this case, the F-coefficients have a simple form:

$$\begin{aligned}
(F^{j_1,\frac{1}{2}})^{m_1,\frac{1}{2}}_{k_1,k_2} &= \delta_{k_1,m_1}\delta_{k_2,\frac{1}{2}} \\
(F^{j_1,\frac{1}{2}})^{m_1,-\frac{1}{2}}_{k_1,k_2} &= \delta_{k_1,m_1}(\delta_{k_2,-\frac{1}{2}} - 2m_1 h\delta_{k_2,\frac{1}{2}}) \\
((F^{-1})^{j_1,\frac{1}{2}})^{n_1,n_2}_{m_1,\frac{1}{2}} &= \delta_{m_1,n_1}(\delta_{n_2,\frac{1}{2}} + 2m_1 h\delta_{n_2,-\frac{1}{2}}) \\
((F^{-1})^{j_1,\frac{1}{2}})^{n_1,n_2}_{m_1,-\frac{1}{2}} &= \delta_{m_1,n_1}\delta_{n_2,-\frac{1}{2}}.
\end{aligned}$$

One can use Wigner's product law, expressed in the form of (3.13) and (3.14), to derive the recurrence relations for $\mathcal{D}^j_{m',m}$ which are reduced to the known recurrence relations of the $SL(2)$ D-functions in the limit of $h = 0$.

Proposition 3.5. *The $SL_h(2)$ D-functions satisfy the following recurrence relations:*

(i) $$\sqrt{j+k}\mathcal{D}^j_{k,m} - (2k-1)h\sqrt{j-k+1}\mathcal{D}^j_{k-1,m} = \sqrt{j+m}\mathcal{D}^{j-\frac{1}{2}}_{k-\frac{1}{2},m-\frac{1}{2}}x + \sqrt{j-m}\mathcal{D}^{j-\frac{1}{2}}_{k-\frac{1}{2},m+\frac{1}{2}}(u-(2m+1)hx)$$

(ii) $$\sqrt{j-k}\mathcal{D}^j_{k,m} = \sqrt{j+m}\mathcal{D}^{j-\frac{1}{2}}_{k+\frac{1}{2},m-\frac{1}{2}}v + \sqrt{j-m}\mathcal{D}^{j-\frac{1}{2}}_{k+\frac{1}{2},m+\frac{1}{2}}(y-(2m+1)hv)$$

(iii) $$\sqrt{j+n}\mathcal{D}^j_{m,n} = \sqrt{j+m}\mathcal{D}^{j-\frac{1}{2}}_{m-\frac{1}{2},n-\frac{1}{2}}(x+(2m-1)hv) + \sqrt{j-m}\mathcal{D}^{j-\frac{1}{2}}_{m+\frac{1}{2},n-\frac{1}{2}}v$$

(iv) $$\sqrt{j-n}\mathcal{D}^j_{m,n} + \sqrt{j+n}(2n+1)h\mathcal{D}^j_{m,n+1} = \sqrt{j+m}\mathcal{D}^{j-\frac{1}{2}}_{m-\frac{1}{2},n+\frac{1}{2}}(u+(2m-1)hy) + \sqrt{j-m}\mathcal{D}^{j-\frac{1}{2}}_{m+\frac{1}{2},n+\frac{1}{2}}y$$

(v) $$\sqrt{j-k+1}\mathcal{D}^j_{k,m} + (2k-1)h\sqrt{j+k}\mathcal{D}^j_{k-1,m} = \sqrt{j-m+1}\mathcal{D}^{j+\frac{1}{2}}_{k-\frac{1}{2},m-\frac{1}{2}}x - \sqrt{j+m+1}\mathcal{D}^{j+\frac{1}{2}}_{k-\frac{1}{2},m+\frac{1}{2}}(u-(2m+1)hx)$$

(vi) $$\sqrt{j+k+1}\mathcal{D}^j_{k,m} = -\sqrt{j-m+1}\mathcal{D}^{j+\frac{1}{2}}_{k+\frac{1}{2},m-\frac{1}{2}}v + \sqrt{j+m+1}\mathcal{D}^{j+\frac{1}{2}}_{k+\frac{1}{2},m+\frac{1}{2}}(y-(2m+1)hv)$$

(vii) $$\sqrt{j-n+1}\mathcal{D}^j_{m,n} = \sqrt{j-m+1}\mathcal{D}^{j+\frac{1}{2}}_{m-\frac{1}{2},n-\frac{1}{2}}(x+(2m-1)hv) - \sqrt{j+m+1}\mathcal{D}^{j+\frac{1}{2}}_{m+\frac{1}{2},n-\frac{1}{2}}v$$

(viii) $$\sqrt{j+n+1}\mathcal{D}^j_{m,n} - \sqrt{j-n}(2n+1)h\mathcal{D}^j_{m,n+1} = -\sqrt{j-m+1}\mathcal{D}^{j+\frac{1}{2}}_{m-\frac{1}{2},n+\frac{1}{2}}(u+(2m-1)hy) + \sqrt{j+m+1}\mathcal{D}^{j+\frac{1}{2}}_{m+\frac{1}{2},n+\frac{1}{2}}y.$$

Proof. Put $j_2 = \frac{1}{2}$, $j = j_1 + \frac{1}{2}$ in relation (3.13), then

$$\begin{aligned}
&\sqrt{j_1+k_1+1}\mathcal{D}^{j_1+\frac{1}{2}}_{k_1+\frac{1}{2},m} + \sqrt{j_1-k_1+1}(\delta_{k_2,-\frac{1}{2}} - 2k_1 h\delta_{k_2,\frac{1}{2}})\mathcal{D}^{j_1+\frac{1}{2}}_{k_1-\frac{1}{2},m} \\
&\quad = \sqrt{j_1+m+\tfrac{1}{2}}\mathcal{D}^{j_1}_{k_1,m-\frac{1}{2}}\mathcal{D}^{\frac{1}{2}}_{k_2,\frac{1}{2}} \\
&\qquad + \sqrt{j_1-m+\tfrac{1}{2}}\mathcal{D}^{j_1}_{k_1,m+\frac{1}{2}}(\mathcal{D}^{\frac{1}{2}}_{k_2,-\frac{1}{2}} - (2m+1)h\mathcal{D}^{\frac{1}{2}}_{k_2,\frac{1}{2}}).
\end{aligned}$$

Replacing $j_1 + \frac{1}{2}$ and $k_1 + \frac{1}{2}$ with j and k, respectively, we obtain

$$\sqrt{j+k}\delta_{k_2,\frac{1}{2}}\mathcal{D}^j_{k,m} + \sqrt{j-k+1}(\delta_{k_2,-\frac{1}{2}} - (2k-1)h\delta_{k_2,\frac{1}{2}})\mathcal{D}^j_{k-1,m}$$
$$= \sqrt{j+m}\mathcal{D}^{j-\frac{1}{2}}_{k-\frac{1}{2},m-\frac{1}{2}}\mathcal{D}^{\frac{1}{2}}_{k_2,\frac{1}{2}} + \sqrt{j-m}\mathcal{D}^{j-\frac{1}{2}}_{k-\frac{1}{2},m+\frac{1}{2}}(\mathcal{D}^{\frac{1}{2}}_{k_2,-\frac{1}{2}} - (2m+1)h\mathcal{D}^{\frac{1}{2}}_{k_2,\frac{1}{2}}).$$

The recurrence relations (i) and (ii) are obtained by putting $k_2 = \frac{1}{2}$ and $k_2 = -\frac{1}{2}$, respectively.

We repeat a similar computation for (3.14). We put $j_2 = \frac{1}{2}$, $j = j_1 + \frac{1}{2}$ in (3.14), then rearrange some variables. We obtain

$$\sqrt{j+n}\{\delta_{n_2,\frac{1}{2}} + (2n-1)h\delta_{n_2,-\frac{1}{2}}\}\mathcal{D}^j_{m,n} + \sqrt{j-n+1}\delta_{n_2,-\frac{1}{2}}\mathcal{D}^j_{m,n-1}$$
$$= \sqrt{j+m}\mathcal{D}^{j-\frac{1}{2}}_{m-\frac{1}{2},n-\frac{1}{2}}\{\mathcal{D}^{\frac{1}{2}}_{\frac{1}{2},n_2} + (2m-1)h\mathcal{D}^{\frac{1}{2}}_{-\frac{1}{2},n_2}\}$$
$$+\sqrt{j-m}\mathcal{D}^{j-\frac{1}{2}}_{m+\frac{1}{2},n-\frac{1}{2}}\mathcal{D}^{\frac{1}{2}}_{-\frac{1}{2},n_2}.$$

The recurrence relations (iii) and (iv) correspond to the cases of $n_2 = \frac{1}{2}$ and $n_2 = -\frac{1}{2}$, respectively.

The recurrence relations (v)–(viii) correspond to $j_2 = \frac{1}{2}$, $j = j_1 - \frac{1}{2}$. In this case, after rearrangement of variables, (3.13) yields

$$\sqrt{j-k+1}\delta_{k_2,\frac{1}{2}}\mathcal{D}^j_{k,m} - \sqrt{j+k}(\delta_{k_2,-\frac{1}{2}} - (2k-1)h\delta_{k_2,\frac{1}{2}})\mathcal{D}^j_{k-1,m}$$
$$= \sqrt{j-m+1}\mathcal{D}^{j+\frac{1}{2}}_{k-\frac{1}{2},m-\frac{1}{2}}\mathcal{D}^{\frac{1}{2}}_{k_2,\frac{1}{2}} - \sqrt{j+m+1}\mathcal{D}^{j+\frac{1}{2}}_{k-\frac{1}{2},m+\frac{1}{2}}$$
$$\times(\mathcal{D}^{\frac{1}{2}}_{k_2,-\frac{1}{2}} - (2m+1)h\mathcal{D}^{\frac{1}{2}}_{k_2,\frac{1}{2}}).$$

Putting $k_2 = \frac{1}{2}$ and $-\frac{1}{2}$, we obtain the relations (v) and (vi), respectively. The relation (3.14) yields

$$\sqrt{j-n+1}(\delta_{n_2,\frac{1}{2}} + (2n-1)h\delta_{n_2,-\frac{1}{2}})\mathcal{D}^j_{m,n} - \sqrt{j+n}\delta_{n_2,-\frac{1}{2}}\mathcal{D}^j_{m,n-1}$$
$$= \sqrt{j-m+1}\mathcal{D}^{j+\frac{1}{2}}_{m-\frac{1}{2},n-\frac{1}{2}}(\mathcal{D}^{\frac{1}{2}}_{\frac{1}{2},n_2} + (2m-1)h\mathcal{D}^{\frac{1}{2}}_{-\frac{1}{2},n_2})$$
$$-\sqrt{j+m+1}\mathcal{D}^{j+\frac{1}{2}}_{m+\frac{1}{2},n-\frac{1}{2}}\mathcal{D}^{\frac{1}{2}}_{-\frac{1}{2},n_2}.$$

The recurrence relations (vii) and (viii) are obtained as the cases of $n_2 = \frac{1}{2}$ and $n_2 = -\frac{1}{2}$, respectively. □

It is possible to obtain the explicit form of D-functions for some special cases such as $\mathcal{D}^j_{m',j}$, $\mathcal{D}^j_{j,m}$ by solving these recurrence relations. However, it seems to be difficult to derive formulae for $\mathcal{D}^j_{m',m}$ for any values of j, m' and m. We will solve this problem by using the tensor operator approach in section 5.

The orthogonality-like relations for $\mathcal{D}^j_{m',m}$ can be obtained from (3.13) and (3.14).

Proposition 3.6. *The D-functions for $SL_h(2)$ $\mathcal{D}^j_{m',m}$ satisfy the orthogonality-like relations which are reduced to the orthogonality relations of $SL(2)$ D-functions in the limit of $h = 0$:*

$$\sum_{m_1,m_2}(-1)^{k_1-m_1}(F^{j,j})^{m_1,-m_1}_{m_1,m_2}\mathcal{D}^j_{k_1,m_1}\mathcal{D}^j_{k_2,m_2} = (F^{j,j})^{k_1,-k_1}_{k_1,k_2} \tag{3.21}$$

$$\sum_{k_1,k_2}(-1)^{m_1-k_1}((F^{-1})^{j,j})^{k_1,k_2}_{k_1,-k_1}\mathcal{D}^j_{k_1,m_1}\mathcal{D}^j_{k_2,m_2} = ((F^{-1})^{j,j})^{m_1,m_2}_{m_1,-m_1}. \tag{3.22}$$

Proof. Consider the cases of $j = 0$, $j_1 = j_2$ in (3.13) and (3.14). Writing $j_1 = j_2 = j$, they yield

$$\sum_{m_1,m_2} \Omega^{j,j,0}_{m_1,m_2,0} \mathcal{D}^j_{k_1,m_1} \mathcal{D}^j_{k_2,m_2} = \Omega^{j,j,0}_{k_1,k_2,0}$$

$$\sum_{k_1,k_2} \mho^{j,j,0}_{k_1,k_2,0} \mathcal{D}^j_{k_1,m_1} \mathcal{D}^j_{k_2,m_2} = \mho^{j,j,0}_{m_1,m_2,0}.$$

The CGCs are given by

$$\Omega^{j,j,0}_{m_1,m_2,0} = \sum_s C^{j,j,0}_{s,-s,0} (F^{j,j})^{s,-s}_{m_1,m_2} \qquad \mho^{j,j,0}_{m_1,m_2,0} = \sum_s C^{j,j,0}_{s,-s,0} ((F^{-1})^{j,j})^{m_1,m_2}_{s,-s}$$

and

$$(F^{j,j})^{s,-s}_{m_1,m_2} = \delta_{s,m_1} \langle j m_2 | \mathrm{e}^{-s\sigma} | j-s \rangle \qquad ((F^{-1})^{j,j})^{m_1,m_2}_{s,-s} = \delta_{s,m_1} \langle j-s | \mathrm{e}^{m_1\sigma} | j m_2 \rangle.$$

Then the proof of proposition 3.6 is straightforward. □

4. Review of $SL(2)$ representation functions

This section is devoted to a review of the D-functions for Lie group $SL(2)$. In particular, we focus on tensor operator properties and the relationship to Jacobi polynomials. We write the D-functions for $SL(2)$ in terms of boson operators. This makes the tensorial properties of D-functions clear.

Let $a_i^j, \bar{a}_i^j, i, j \in \{1, 2\}$ be four copies of a boson operator commuting with one another, i.e.

$$[\bar{a}_i^j, a_k^\ell] = \delta_{i,k}\delta^{j,\ell} \qquad [a_i^j, a_k^\ell] = [\bar{a}_i^j, \bar{a}_k^\ell] = 0. \tag{4.1}$$

It is known that the Lie algebra $gl(2) \oplus gl(2)$ is realized by these boson operators. The left (lower) generators are defined by

$$E_{ij} = a_i^1 \bar{a}_j^1 + a_i^2 \bar{a}_j^2 \tag{4.2}$$

and the right (upper) generators are defined by

$$E^{ij} = a_1^i \bar{a}_1^j + a_2^i \bar{a}_2^j. \tag{4.3}$$

Then both left and right generators satisfy the $gl(2)$ commutation relations and, furthermore, $[E_{ij}, E^{k,\ell}] = 0$. Each $gl(2)$ has decomposition $gl(2) = sl(2) \oplus u(1)$. The left and right $sl(2)$ are generated by

$$J_+ = E_{21} \qquad J_- = E_{12} \qquad J_0 = E_{22} - E_{11} \tag{4.4}$$

and

$$K_+ = E^{12} \qquad K_- = E^{21} \qquad K_0 = E^{11} - E^{22} \tag{4.5}$$

respectively, and $u(1)$ sectors by $Z_L = -E_{11} - E_{22}$ and $Z_R = E^{11} + E^{22}$. This choice of generators may be different from the usual one (see, for example, [6, section 4.4]). However, it is a suitable choice for twisting discussed in the next section. Note also that, in this realization, $Z_L = -Z_R$. Therefore, strictly speaking, this realization is not the direct sum of two copies of $gl(2)$.

The D-functions for Lie group $GL(2)$ can be given in terms of a_i^j:

$$\mathcal{D}^{(0)j}_{m',m} = \{(j+m')!(j-m')!(j+m)!(j-m)!\}^{1/2} \sum_{K,L,M,N} \frac{(a_1^1)^K (a_2^1)^L (a_1^2)^M (a_2^2)^N}{K!L!M!N!} \tag{4.6}$$

where the sum over K, L, M and N runs non-negative integers provided that

$$K + L = j + m \qquad M + N = j - m$$
$$K + M = j + m' \qquad L + N = j - m'. \tag{4.7}$$

We obtain $SL(2)$ D-functions by imposing $a^1_1 a^2_2 - a^1_2 a^2_1 = 1$.

It is not difficult to see that D-functions (4.6) form the irreducible tensor operators for both left and right $gl(2)$, i.e.

$$[J_\pm, \mathcal{D}^{(0)j}_{m',m}] = \sqrt{(j \pm m')(j \mp m' + 1)}\mathcal{D}^{(0)j}_{m'\mp 1,m}$$
$$[J_0, \mathcal{D}^{(0)j}_{m',m}] = -2m'\mathcal{D}^{(0)j}_{m',m} \qquad [Z_L, \mathcal{D}^{(0)j}_{m',m}] = -2j\mathcal{D}^{(0)j}_{m',m} \tag{4.8}$$

and

$$[K_\pm, \mathcal{D}^{(0)j}_{m',m}] = \sqrt{(j \mp m)(j \pm m + 1)}\mathcal{D}^{(0)j}_{m',m\pm 1}$$
$$[K_0, \mathcal{D}^{(0)j}_{m',m}] = 2m\mathcal{D}^{(0)j}_{m',m} \qquad [Z_R, \mathcal{D}^{(0)j}_{m',m}] = 2j\mathcal{D}^{(0)j}_{m',m}. \tag{4.9}$$

It is well known that the D-functions for $SL(2)$ can be expressed in terms of Jacobi polynomials. The Jacobi polynomials are defined by

$$P^{(\alpha,\beta)}_n(z) = \sum_{r \geqslant 0} \frac{(-n)_r(\alpha + \beta + n + 1)_r}{(1)_r(\alpha + 1)_r} z^r \tag{4.10}$$

where $(\alpha)_r$ stands for the sifted factorial

$$(\alpha)_r = \alpha(\alpha + 1)\cdots(\alpha + r - 1).$$

For the case of $SL(2)$, we have the relation $a^1_1 a^2_2 = 1 + a^1_2 a^2_1$. Using this, the D-functions are expressed for $m' + m \geqslant 0$, $m' \geqslant m$:

$$\mathcal{D}^{(0)j}_{m',m} = \left\{ \binom{j + m'}{m' - m} \binom{j - m}{m' - m} \right\}^{1/2} (a^1_1)^{m'+m}(a^2_1)^{m'-m} P^{(m'-m,m'+m)}_{j-m'}(z) \tag{4.11}$$

where $z \equiv -a^1_2 a^2_1$. We have similar relations for other cases.

5. Representation functions for $SL_h(2)$

5.1. Explicit formulae for D-functions

We saw, in the previous section, that the D-functions for $GL(2)$ form the irreducible tensor operators of both left and right $gl(2)$. This fact leads us to the expectation that the D-functions for $GL_h(2)$ also form the irreducible tensor operators of left and right $\mathcal{U}_h(gl(2))$. It is known that the tensor operators for $\mathcal{U}_h(gl(2))$ can be obtained from the ones for $gl(2)$ by twisting [11,25]. Therefore, we may obtain the D-functions for $GL_h(2)$ from the one for $GL(2)$ by twisting twice. The irreducible tensor operators for $\mathcal{U}_h(gl(2))$ are defined by replacing the commutator in the LHS of (4.8) and (4.9) with the adjoint action. Let $\boldsymbol{t}$ be a any tensor operator for $\mathcal{U}_h(gl(2))$ and $X \in \mathcal{U}_h(gl(2))$, then the adjoint action of X on $\boldsymbol{t}$ is defined by [26]

$$\mathrm{ad}X(\boldsymbol{t}) = m(\mathrm{id} \otimes S)(\Delta(X)(\boldsymbol{t} \otimes 1)). \tag{5.1}$$

The tensor operators $\boldsymbol{t}$ for $\mathcal{U}_h(gl(2))$ and the tensor operators $\boldsymbol{t}^{(0)}$ for $gl(2)$ are related via the twist element $\mathcal{F}$ by ([25], see also [11])

$$\boldsymbol{t} = m(\mathrm{id} \otimes S)(\mathcal{F}(\boldsymbol{t}^{(0)} \otimes 1)\mathcal{F}^{-1}). \tag{5.2}$$

Note that $gl(2)$ and $\mathcal{U}_h(gl(2))$ have the same commutation relations so that the realization (4.2), (4.3) is the realization of $\mathcal{U}_h(gl(2))$ as well. We consider the tensor operators under this realization of $\mathcal{U}_h(gl(2))$.

Let us first consider the simplest case: $j = \frac{1}{2}$. What we obtain in this case from (4.6), (4.8) and (4.9) is that the pairs (a_1^1, a_2^1), (a_1^2, a_2^2) are spinors of the left $gl(2)$ and the pairs (a_1^1, a_1^2), (a_2^1, a_2^2) are spinors of the right $gl(2)$. Namely, each boson operator a_i^j is a component of spinor for both left and right $gl(2)$. This fact tells us that, by twisting via the elements

$$\mathcal{F}_L = \exp(-\tfrac{1}{2} J_0 \otimes \sigma_L) \qquad \mathcal{F}_R = \exp(-\tfrac{1}{2} K_0 \otimes \sigma_R) \tag{5.3}$$

with $\sigma_L = -\ln(1 - 2hJ_+)$, $\sigma_R = -\ln(1 - 2hK_+)$, we obtain a element of spinor for both left and right $\mathcal{U}_h(sl(2))$. To this end, it is convenient to rewrite (5.2) in a different form. Let us write the twist element and its inverse as

$$\mathcal{F} = \sum_a f^a \otimes f_a \qquad \mathcal{F}^{-1} = \sum_a g^a \otimes g_a$$

then

$$\mu = \sum_a f^a S_0(f_a) \qquad \mu^{-1} = \sum_a S_0(g^a) g_a.$$

Noting the identity

$$\sum g^b \mu S_0(g_b) = \sum g^b f^a S_0(g_b f_a) = m(\mathrm{id} \otimes S_0)(\mathcal{F}^{-1}\mathcal{F}) = 1$$

the relation (5.2) yields

$$\boldsymbol{t} = \sum f^a \boldsymbol{t}^{(0)} g^b S(f_a g_b) = \sum f^a \boldsymbol{t}^{(0)} \mu S_0(f_a g_b) \mu^{-1} = \sum f^a \boldsymbol{t}^{(0)} S_0(f_a) \mu^{-1}. \tag{5.4}$$

From (5.4), the twisting by $\mathcal{F}_L$ reads

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{2}\right)^n J_0^n a_i^j S_0(\sigma_L) \mu^{-1} &= a_i^j \sum_{k=0}^{\infty} \frac{(-1)^{ik}}{k!} \left(-\frac{1}{2}\right)^k S(\sigma_L)^k \\ &= a_i^j \exp\{(-1)^i \sigma_L / 2\}. \end{aligned}$$

We used the fact that $S(\sigma_L) = -\sigma_L$ in the last equality. To twist the above-obtained result by $\mathcal{F}_R$, we can repeat a similar computation. Then we have the doubly twisted boson operators

$$a_i^j \exp\{(-1)^i \sigma_L/2 + (-1)^{j+1} \sigma_R/2\}. \tag{5.5}$$

The commutation relations of the twisted boson operators (5.5) are obtained by straightforward computation, showing that the twisted boson operators give a realization of the generators of $GL_h(2)$.

Proposition 5.1. *Let*

$$\begin{aligned} x &= a_1^1 \mathrm{e}^{(-\sigma_L + \sigma_R)/2} \qquad & u &= a_1^2 \mathrm{e}^{-(\sigma_L + \sigma_R)/2} \\ v &= a_2^1 \mathrm{e}^{(\sigma_L + \sigma_R)/2} \qquad & y &= a_2^2 \mathrm{e}^{(\sigma_L - \sigma_R)/2} \end{aligned} \tag{5.6}$$

then, x, u, v and y satisfy the commutation relations of the generators of $GL_h(2)$ (2.1). In this realization, the central element D is given by

$$D \equiv xy - uv - hxv = a_1^1 a_2^2 - a_2^1 a_1^2. \tag{5.7}$$

Note that the central element D remains undeformed in this realization.

Proof. One can verify the commutation relations directly. Here we give some useful commutation relations for verification: the commutation relations between σ_L, σ_R and boson operators,

$$\begin{aligned} [\sigma_L, a_1^1] &= 2h\mathrm{e}^{\sigma_L} a_2^1 \qquad & [\sigma_L, a_1^2] &= 2h\mathrm{e}^{\sigma_L} a_2^2 \\ [\sigma_R, a_1^2] &= 2h\mathrm{e}^{\sigma_R} a_1^1 \qquad & [\sigma_R, a_2^2] &= 2h\mathrm{e}^{\sigma_R} a_2^1. \end{aligned}$$

These are easily verified by using the power series expansion of σ_L, σ_R: $\sigma_L = \sum_{n=1}^{\infty} \frac{(2hJ_+)^n}{n}$. These relations can be used to prove the following commutation relations which hold for any real k:

$$\begin{aligned} &[\mathrm{e}^{k\sigma_L}, a_1^1] = 2hk\mathrm{e}^{(k+1)\sigma_L} a_2^1 && [\mathrm{e}^{k\sigma_L}, a_1^2] = 2hk\mathrm{e}^{(k+1)\sigma_L} a_2^2 \\ &[\mathrm{e}^{k\sigma_R}, a_1^2] = 2hk\mathrm{e}^{(k+1)\sigma_R} a_1^1 && [\mathrm{e}^{k\sigma_R}, a_2^2] = 2hk\mathrm{e}^{(k+1)\sigma_R} a_2^1. \end{aligned} \tag{5.8}$$

□

Next let us consider the twisting of $\mathcal{D}^{(0)j}_{m',m}$ for any values of j by the twist elements $\mathcal{F}_L, \mathcal{F}_R$. We denote the doubly twisted $\mathcal{D}^{(0)j}_{m',m}$ by $\mathcal{D}^{j}_{m',m}$, since it will be shown later that this $\mathcal{D}^{j}_{m',m}$ gives the D-functions for $GL_h(2)$. The computation is almost the same as for the case of spinors. What we need to compute is the twisting of $(a_1^1)^K(a_2^1)^L(a_1^2)^M(a_2^2)^N$ in expression (4.6). The twisting by $\mathcal{F}_L$ reads

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{1}{2}\right)^n J_0^n (a_1^1)^K(a_2^1)^L(a_1^2)^M(a_2^2)^N S_0^n(\sigma_L)\mu^{-1} \\ &\qquad = (a_1^1)^K(a_2^1)^L(a_1^2)^M(a_2^2)^N \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{1}{2}\right)^k (-K+L-M+N)^k \mu S_0^k(\sigma_L)\mu^{-1} \\ &\qquad = (a_1^1)^K(a_2^1)^L(a_1^2)^M(a_2^2)^N \exp\{-(K-L+M-N)\sigma_L/2\}. \end{aligned}$$

Further twisting by $\mathcal{F}_R$ gives

$$(a_1^1)^K(a_2^1)^L(a_1^2)^M(a_2^2)^N \exp\{-(K-L+M-N)\sigma_L/2 + (K+L-M-N)\sigma_R/2\}. \tag{5.9}$$

Because of (4.7), we have $K-L+M-N = 2m'$ and $K+L-M-N = 2m$. Thus, the exponential factor in (5.9) is factored out of the sum over K, L, M and N. Therefore, we have proved the following proposition.

Proposition 5.2. *In the realization (4.2), (4.3), the irreducible tensor operators of both left and right* $\mathcal{U}_h(gl(2))$ *are given by*

$$\mathcal{D}^{j}_{m',m} = \mathcal{D}^{(0)j}_{m',m}\mathrm{e}^{-m'\sigma_L + m\sigma_R}. \tag{5.10}$$

One can write $\mathcal{D}^{j}_{m',m}$ of proposition 5.2 in terms of the generators of $GL_h(2)$ by making use of proposition 5.1. For real A, B,

$$\begin{aligned} (a_1^1)^K \mathrm{e}^{(A\sigma_L+B\sigma_R)/2} &= (a_1^1)^{K-1} x \mathrm{e}^{(A+1)\sigma_L/2+(B-1)\sigma_R/2} \\ &= (a_1^1)^{K-1}\mathrm{e}^{(A+1)\sigma_L/2+(B-1)\sigma_R/2}\{\mathrm{e}^{-(A+1)\sigma_L/2} x \mathrm{e}^{(A+1)\sigma_L/2}\}. \end{aligned}$$

The expression $\{\cdots\}$ in the last line can be calculated by using (5.8) and gives $x - h(A+1)v$. Thus, we obtain

$$\begin{aligned} (a_1^1)^K \mathrm{e}^{(A\sigma_L+B\sigma_R)/2} &= \mathrm{e}^{(A+K)\sigma_L/2+(B-K)\sigma_R/2} \\ &\quad \times (x - h(A+K)v)(x - h(A+K-1)v)\cdots(x - h(A+1)v). \end{aligned} \tag{5.11}$$

Similar computation gives three other identities:

$$\begin{aligned} (a_2^1)^L \mathrm{e}^{(A\sigma_L+B\sigma_R)/2} &= \mathrm{e}^{(A-L)\sigma_L/2+(B-L)\sigma_R/2} v^L \\ (a_1^2)^M \mathrm{e}^{(A\sigma_L+B\sigma_R)/2} &= \mathrm{e}^{(A+M)\sigma_L/2+(B+M)\sigma_R/2} \\ &\quad \times (u - h(B+M)x - h(A+M)y + h^2(A+M)(B+M)v) \\ &\quad \times (u - h(B+M-1)x - h(A+M-1)y \\ &\quad + h^2(A+M-1)(B+M-1)v) \\ &\quad \times \cdots \times (u - h(B+1)x - h(A+1)y + h^2(A+1)(B+1)v) \\ (a_2^2)^N \mathrm{e}^{(A\sigma_L+B\sigma_R)/2} &= \mathrm{e}^{(A-N)\sigma_L/2+(B+N)\sigma_R/2} \\ &\quad \times (y - h(B+N)v)(h - h(B+N-1)v)\cdots(y - h(B+1)v). \end{aligned} \tag{5.12}$$

The boson operators a_i^j commute with one another so that the order of a_i^j in $\mathcal{D}^{(0)j}_{m',m}$ is irrelevant. Therefore, we can have different expressions of $\mathcal{D}^j_{m',m}$ depending on the choice of the order of boson operators. Here we give two of them, and show that they are the representation functions of $GL_h(2)$.

Proposition 5.3. *The D-function for $GL_h(2)$ are given by*

$$\mathcal{D}^j_{m',m} = \{(j+m')!(j-m')!(j+m)!(j-m)!\}^{1/2} \times \sum_{K,L,M,L} \frac{X_K v^L U_{K,L,M} Y_{K,L,M,N}}{K!M!L!N!} \tag{5.13}$$

where X_K, $U_{K,L,M}$ and $Y_{K,L,M,N}$ are defined by

$$\begin{aligned}
X_K &= x(x+hv)\cdots(x+h(K-1)v)\\
U_{K,L,M} &= (u-h(K+L)x+h(K-L)y-h^2(K^2-L^2)v)\\
&\quad\times(u-h(K+L-1)x+h(K-L+1)y-h^2(K^2-(L-1)^2)v)\\
&\quad\times\cdots\times(u-h(K+L-M+1)x+h(K-L+M-1)y\\
&\quad-h^2(K^2-(L-M+1)^2)v)\\
Y_{K,L,M,N} &= (y-h(K+L-M)v)(y-h(K+L-M-1)v)\\
&\quad\times\cdots(y-h(K+L-M-N+1)v).
\end{aligned}$$

The D-functions have another expression which is

$$\mathcal{D}^j_{m',m} = \{(j+m')!(j-m')!(j+m)!(j-m)!\}^{1/2} \sum_{K,L,M,L} \frac{U_M X_{K,M} Y_{K,M,N} v^L}{K!M!L!N!} \tag{5.14}$$

where U_M, $X_{K,M}$, $Y_{K,M,N}$ are defined by

$$\begin{aligned}
U_M &= u(u+h(x+y)+h^2v)\cdots(u+h(M-1)(x+y)+h^2(M-1)^2v)\\
X_{K,M} &= (x+hMv)(x+h(M+1)v)\cdots(x+h(K+M-1)v)\\
Y_{K,M,N} &= (y-h(K-M)v)(y-h(K-M-1)v)\cdots(y-h(K-M-N+1)v)v^L.
\end{aligned}$$

The sum over K, L, M and N runs non-negative integers under the condition (4.7).

Remark. We obtain the D-functions for $SL_h(2)$ by putting $D = xy - uv - hxv = 1$.

Proof. These expressions are obtained by using (5.11) and (5.12). Expression (5.13) corresponds to the boson ordering $(a_1^1)^k(a_2^1)^L(a_1^2)^M(a_2^2)^N$, while (5.14) corresponds to $(a_1^2)^M(a_1^1)^K(a_2^2)^N(a_2^1)^L$.

To show that $\mathcal{D}^j_{m',m}$ are the representation functions of $GL_h(2)$, we must verify (2.8). It is obvious that $\mathcal{D}^j_{m',m} \in GL_h(2)$ and the co-unit of $\mathcal{D}^j_{m',m}$ is easily verified by using $\epsilon(x) = \epsilon(y) = 1$, $\epsilon(u) = \epsilon(v) = 0$. However, it seems to be difficult to verify the coproduct of $\mathcal{D}^j_{m',m}$ by straightforward computation. Instead of verifying the coproduct, we show that $\mathcal{D}^j_{m',m}$ satisfy the recurrence relations of proposition 3.5. Note that the recurrence relations of proposition 3.5 are for $SL_h(2)$. The Jordanian deformation of the Lie algebra $gl(2)$ considered in this paper is the direct sum of the deformed $sl(2)$ and undeformed $u(1)$: $\mathcal{U}_h(gl(2)) = \mathcal{U}_h(sl(2)) \oplus u(1)$. This implies that the CGCs for $\mathcal{U}_h(sl(2))$ also give the CGCs for $\mathcal{U}_h(gl(2))$. Therefore, the D-functions for $GL_h(2)$ also satisfy the recurrence relations of proposition 3.5.

As an example, we show that the $\mathcal{D}^j_{m',m}$ give the solutions to the recurrence relation (ii) of proposition 3.5. We substitute expression (5.14) of the D-functions into the first term of the RHS of (ii), then replace the dummy index L with $L-1$. It follows that

$$\sqrt{j+m}\mathcal{D}^{j-\frac{1}{2}}_{k-\frac{1}{2},m-\frac{1}{2}}v = \{(j+m)!(j-m)!(j+k-1)!(j-k)!\}^{1/2} \times \sum_{K,L,M,N} L\frac{U_M X_{K,M} Y_{K,M,N} v^L}{K!M!L!N!}$$

where the indices K, L, M and N satisfy the condition

$$\begin{aligned} K+L &= j+m \qquad & M+N &= j-m \\ K+M &= j+k-1 \qquad & L+N &= j-k+1. \end{aligned} \tag{5.15}$$

For the second term in the RHS of (ii), we use (5.13). Replacing the index N with $N-1$, we obtain

$$\begin{aligned} &\sqrt{j-m}\mathcal{D}^{j-\frac{1}{2}}_{k-\frac{1}{2},m+\frac{1}{2}}(u-(2m+1)hv) \\ &\quad = \{(j+m)!(j-m)!(j+k-1)!(j-k)!\}^{1/2} \times \sum_{K,L,M,N} N\frac{X_K v^L U_{K,L,M} Y_{K,L,M,N}}{K!M!L!N!} \end{aligned}$$

where the indices K, L, M and N also satisfy (5.15). Since the expressions (5.13) and (5.14) are different expressions of the same D-functions, it holds that $U_M X_{K,M} Y_{K,M,N} v^L = X_K v^L U_{K,L,M} Y_{K,L,M,N}$. Therefore, the RHS of (ii) reads

$$\begin{aligned} &\{(j+m)!(j-m)!(j+k-1)!(j-k)!\}^{1/2} \sum_{K,L,M,N} (L+N)\frac{X_K v^L U_{K,L,M} Y_{K,L,M,N}}{K!M!L!N!} \\ &\quad = \sqrt{j-k+1}\mathcal{D}^j_{k-1,m}. \end{aligned}$$

The four-term recurrence relation (i) of proposition 3.5 is reduced to a three-term relation, by eliminating $\mathcal{D}^j_{k-1,m}$ from (i) and (ii). This recurrence relation is easily solved by using another expression of $\mathcal{D}^j_{k,m}$ corresponding to another ordering of boson operators. The suitable expressions for solving it are the ones obtained from the ordering $(a_1^2)^M(a_2^2)^N(a_2^1)^L(a_1^1)^K$ and $(a_1^1)^K(a_2^2)^N(a_2^1)^L(a_1^2)^M$. In this way, we can verify that the $\mathcal{D}^j_{m',m}$ obtained in this proposition solve all the recurrence relations given in proposition 3.5. □

Both (5.13) and (5.14), of course, give the generators of $GL_h(2)$ for $j=\frac{1}{2}$ which reflects proposition 3.4. The D-functions for $j=1$ read

$$\mathcal{D}^1 = \begin{pmatrix} x^2+hxv & \sqrt{2}(ux+huv) & u^2+h(ux+uy+huv) \\ \sqrt{2}xv & D+2uv & \sqrt{2}(uy+huv) \\ v^2 & \sqrt{2}yv & y^2+hyv \end{pmatrix}. \tag{5.16}$$

For $SL_h(2)$, i.e. putting $D=1$, this coincides with the expression obtained by using the h-symplecton or quantum h-plane [11]. Chakrabarti and Quesne obtained the $\mathcal{D}^1$ for two-parametric Jordanian deformation of $GL(2)$ in the coloured representation through a contraction technique to the D-functions for standard (q,λ)-deformation of $GL(2)$ [9]. To compare the present $\mathcal{D}^1$ with the one given in [9], put $\alpha=0, z=1$ in equations (4.20) and (4.21) of [9]. Then we see that the D-functions for $j=1$ of [9] are different from (5.16). This difference stems from the different choice of the basis of $\mathcal{U}_h(sl(2))$. In [9], the basis introduced by Ohn [27] is used, that is, the commutation relations of the generators of $\mathcal{U}_h(sl(2))$ are not the same as those of $sl(2)$, while the basis of this paper satisfies the same commutation relations as $sl(2)$. This results in different CGCs for the same algebra so that the recurrence relations for the D-functions have different form. The CGCs for Ohn's basis are found in [20]. Repeating the same procedure as in section 3.2, we obtain another form of recurrence relations. It should be easy to verify that the $\mathcal{D}^1$ of [9] solves these recurrence relations.

5.2. $SL_h(2)$ D-Functions and Jacobi polynomials

The purpose of this section is to show that the D-functions for $SL_h(2)$ can be expressed in terms of Jacobi polynomials. To this end, we return to the boson realization of D-functions (proposition 5.2) and use the fact that the D-functions for Lie group $SL(2)$ are written in terms of Jacobi polynomials. Recall the following two facts: (1) the central element D of $GL_h(2)$ is not deformed in the boson realization (5.7), (2) Jacobi polynomials in the D-functions for $SL(2)$ are power series in the variable $z = -a_2^1 a_1^2$. We write the D-functions $\mathcal{D}^{(0)j}_{m',m}$ for $SL(2)$ in (5.10) in terms of Jacobi polynomials and then use the easily proved relation $(a_2^1 a_1^2)^r = (uv)^r$ in order to replace the variable $z = -a_2^1 a_1^2$ with the h-deformed one $z = -uv$. Let us consider, as an example, the case of $m' + m \geqslant 0, m' \geqslant m$. The $\mathcal{D}^{(0)j}_{m',m}$ are given by (4.11). We rearrange the order of a_1^1, a_1^2 and $P^{(m'-m,m'+m)}_{j-m'}(z)$ to be $P^{(m'-m,m'+m)}_{j-m'}(z)(a_1^2)^{m'-m}(a_1^1)^{m'+m}$. Using (5.11) and (5.12), we see that

$$
\begin{aligned}
&(a_1^2)^{m'-m}(a_1^1)^{m'+m}\mathrm{e}^{-m'\sigma_L+m\sigma_R} \\
&\qquad = u(u+h(x+y)+h^2v)\cdots(u+h(m'-m-1)(x+y)+h^2(m'-m-1)^2v) \\
&\qquad\quad \times(x+h(m'-m)v)(x+h(m'-m-1)v)\cdots(x+h(2m'-1)v).
\end{aligned}
$$

This completes the expression of D-functions in terms of Jacobi polynomials.

Repeating this process for other cases, we can prove the next proposition.

Proposition 5.4. *The D-functions for $SL_h(2)$ are written in terms of Jacobi polynomials as follows:*

(i) $m' + m \geqslant 0, m' \geqslant m$

$$
\begin{aligned}
\mathcal{D}^j_{m',m} = N_+ P^{(m'-m,m'+m)}_{j-m'}(z)
&\times u(u+h(x+y)+h^2v)\cdots(u+h(m'-m-1)(x+y) \\
&+h^2(m'-m-1)^2v) \\
&\times(x+h(m'-m)v)(x+h(m'-m-1)v)\cdots(x+h(2m'-1)v).
\end{aligned}
$$

(ii) $m' + m \geqslant 0, m' \leqslant m$

$$
\mathcal{D}^j_{m',m} = N_- P^{(-m'+m,m'+m)}_{j-m}(z)x(x+hv)\cdots(x+h(m'+m-1)v)v^{-m'+m}.
$$

(iii) $m' + m \leqslant 0, m' \geqslant m$

$$
\begin{aligned}
\mathcal{D}^j_{m',m} = N_+ P^{(m'-m,-m'-m)}_{j+m}(z)
&\times u(u+h(x+y)+h^2v)\cdots(u+h(m'-m-1)(x+y) \\
&+h^2(m'-m-1)^2v) \\
&\times(y-h(m-m')v)(y-h(m-m'-1)v)\cdots(y-h(2m+1)v).
\end{aligned}
$$

(iv) $m' + m \leqslant 0, m' \leqslant m$

$$
\begin{aligned}
\mathcal{D}^j_{m',m} = N_- P^{(-m'+m,-m'-m)}_{j+m'}(z)
&\times v^{-m'+m}(y-h(m-m')v)(y-h(m-m'-1)v)\cdots(y-h(2m+1)v).
\end{aligned}
$$

The variable z is defined by $z = -uv$ and the factors N_+, N_- by

$$N_+ = \left\{ \binom{j+m'}{m'-m} \binom{j-m}{m'-m} \right\}^{1/2} \qquad N_- = \left\{ \binom{j-m'}{m-m'} \binom{j+m}{m-m'} \right\}^{1/2}.$$

Remark. The Jacobi polynomials are to the left of the generators of $SL_h(2)$. To move $P_n^{(\alpha,\beta)}(z)$ to the right, the relation

$$(uv)^r \exp(-m'\sigma_L + m\sigma_R) = \exp(-m'\sigma_L + m\sigma_R)\{uv - 2h(-m'yv + mxv) - 4h^2 mm'v^2\}^r$$

is used and we see that the Jacobi polynomials are changed to the power series in $\zeta_{m',m} = -(u + 2h(m'y - mx) - 4h^2 mm')v$, but the rest of the formulae remain unchanged.

6. Boson realization of $GL_{h,g}(2)$

It is natural to generalize the results in the previous section to the two-parametric Jordanian deformation of $GL(2)$ [28], since the twist element which generates the two-parametric Jordanian quantum algebra $\mathcal{U}_{h,g}(gl(2))$ [29, 30] is known [31]. Unfortunately, the method of the previous sections leads us to quite complex calculations. As the first step to obtaining the D-functions for two-parametric Jordanian quantum group $GL_{h,g}(2)$, we here give the boson realization of the generators of $GL_{h,g}(2)$.

The left and right twist elements are given by

$$\mathcal{F}_L = \exp\left(\frac{g}{2h}\sigma_L \otimes Z_L\right) \exp\left(-\frac{1}{2}J_0 \otimes \sigma_L\right)$$
$$\mathcal{F}_R = \exp\left(\frac{g}{2h}\sigma_R \otimes Z_R\right) \exp\left(-\frac{1}{2}K_0 \otimes \sigma_R\right)$$

respectively. We can see that the $GL_{h,g}(2)$ is reduced to $GL_h(2)$ when $g = 0$. Repeating the same procedure as (5.5), we obtain the twisted boson operators. We can rewrite the twisted boson operators in terms of the generators $GL_h(2)$. The next proposition can be regarded as a realization of $GL_{h,g}(2)$ by generators of $GL_h(2)$ and Z_L, Z_R as well.

Proposition 6.1. *Let*

$$\begin{aligned} &a = x - gvZ_L \qquad b = u - gxZ_R - gyZ_L + g^2 vZ_L Z_R \\ &c = v \qquad d = y - gvZ_R \end{aligned} \tag{6.1}$$

where x, u, v and y are given by (5.6). Then a, b, c and d satisfy the commutation relation of $GL_{h,g}(2)$.

Remark. In this realization, the quantum determinants $D' = ad - bc - (h+g)ac$ for $GL_{h,g}(2)$ and D for $GL_h(2)$ coincide: $D' = D = a_1^1 a_2^2 - a_2^1 a_1^2$.

Proof. The proof requires lengthy calculation, but is straightforward. The following commutation relations [28] are verified:

$$\begin{aligned} &[a, b] = -(h+g)(D' - a^2) \qquad [a, c] = -(h-g)c^2 \\ &[a, d] = (h+g)ac - (h-g)dc \qquad [b, c] = -(h+g)ac - (h-g)cd \\ &[b, d] = (h-g)(D' - d^2) \qquad [c, d] = (h+g)c^2. \end{aligned} \tag{6.2}$$

□

7. Concluding remarks

In this paper, the explicit formulae of the D-functions for $SL_h(2)$ (and $GL_h(2)$) have been obtained by using the tensor operator technique. We used the fact that the D-functions for Lie group $GL(2)$ form irreducible tensor operators of $gl(2) \oplus gl(2)$ in the realization (4.2), (4.3). This kind of tensor operator is called a double irreducible tensor operator in the literature. The D-functions for $GL_h(2)$ were obtained via the construction of double irreducible tensor operators for $\mathcal{U}_h(gl(2)) \oplus \mathcal{U}_h(gl(2))$. Other examples of double irreducible tensor operators were considered for q-deformation [32, 33] and for Jordanian deformation [34]. Quesne constructed the $GL_h(n) \times GL_{h'}(m)$ covariant bosonic and fermionic algebra which form the double irreducible tensor operators of $\mathcal{U}_h(gl(n)) \oplus \mathcal{U}_{h'}(gl(m))$ using the contraction method [34]. This suggests, in the case of $n = m = 2$ and $h = h'$, that the bosonic algebra of Quesne has a close relation to $\mathcal{D}^{\frac{1}{2}}_{m',m}$, i.e. the generators of $GL_h(2)$.

We also showed that the D-functions for $SL_h(2)$ can be expressed in terms of Jacobi polynomials. Contrary to the q-deformed case where the little q-Jacobi polynomials appear in the D-functions for $SU_q(2)$, the ordinary Jacobi polynomials are associated with the D-functions for $SL_h(2)$. It seems to be a general feature of Jordanian deformation that the ordinary orthogonal polynomials are associated with the representations. It is known that the ordinary Gauss hypergeometric functions are associated with the h-symplecton [11], while the q-hypergeometric functions are associated with q-deformation of the symplecton.

The extension of the results of this paper to the Jordanian deformation of $SL(n)$ should be possible, since the explicit expressions for the twist element are known for the Lie algebra $sl(n)$ [35].

References

[1] Kupershmidt B A 1992 *J. Phys. A: Math. Gen.* **25** L1239

[2] Reshetikhin N Yu, Takhtadzhyan L A and Faddeev L D 1990 *Leningrad Math. J.* **1** 193

[3] Demidov E E *et al* 1990 *Prog. Theor. Phys. Suppl.* **102** 203

[4] Zakrewski S 1991 *Lett. Math. Phys.* **22** 287

[5] Ewen H, Ogievetsky O and Wess J 1991 *Lett. Math. Phys.* **22** 297

[6] Biedenharn L C and Lohe M A 1995 *Quantum Group Symmetry and q-Tensor Algebras* (Singapore: World Scientific)

[7] Chaichian M and Demichev A 1996 *Introduction to Quantum Groups* (Singapore: World Scientific)

[8] Karimipour V 1994 *Lett. Math. Phys.* **30** 87
Karimipour V 1995 *Lett. Math. Phys.* **35** 303
Cho S, Madore J and Park K S 1998 *J. Phys. A: Math. Gen.* **31** 2639
Madore J and Steinacker H 2000 *J. Phys. A: Math. Gen.* **33** 327

[9] Chakrabarti R and Quesne C 1999 *Int. J. Mod. Phys.* A **14** 2511

[10] Aghamohammadi A 1993 *Mod. Phys. Lett.* A **8** 2607

[11] Aizawa N 1999 *J. Math. Phys.* **40** 5921

[12] Vaksman L L and Soibel'man Ya S 1988 *Funct. Anal. Appl.* **22** 170
Koornwinder T H 1989 *Indag. Math.* **51** 97

[13] Groza V A, Kacurik I I and Klimyk A U 1990 *J. Math. Phys.* **31** 2769

[14] Nomura M 1990 *J. Phys. Soc. Japan* **59** 4260

[15] Masuda T *et al* 1991 *J. Funct. Anal.* **99** 357

[16] Nomura M 1991 *J. Phys. Soc. Japan* **60** 710

[17] Drinfeld V G 1990 *Leningrad Math. J.* **1** 1419

[18] Ogievetsky O V 1993 *Proc. Winter School on Geometry and Physics (Zidkov), Suppl. Rendiconti cir. Math. Palermo, Serie II* N **37** 185

[19] Aizawa N 1997 *J. Phys. A: Math. Gen.* **30** 5981

[20] Van der Jeugt J 1998 *J. Phys. A: Math. Gen.* **31** 1495
Czech 1997 *J. Physique* **47** 1283

[21] Drinfeld V G 1987 Quantum groups *Proc. Int. Congress of Math. (Berkeley, 1986)* vol 1, ed A V Gleason, p 798
[22] Fronsdal C and Galindo A 1993 *Lett. Math. Phys.* **27** 59
[23] Jagannathan R and Van der Jeugt J 1995 *J. Phys. A: Math. Gen.* **28** 2819
[24] Bonechi F *et al* 1994 *J. Phys. A: Math. Gen.* **27** 1307
[25] Fiore G 1998 *J. Math. Phys.* **39** 3437
[26] Rittenberg V and Scheunert M 1992 *J. Math. Phys.* **33** 3636
[27] Ohn Ch 1992 *Lett. Math. Phys.* **25** 85
[28] Aghamohammadi A 1993 *Mod. Phys. Lett.* A **8** 2607
[29] Aneva B L, Dobrev V K and Mihov S G 1997 *J. Phys. A: Math. Gen.* **30** 6769
[30] Parashar P 1998 *Lett. Math. Phys.* **45** 105
[31] Aizawa N 1998 *Czech. J. Phys.* **48** 1273
[32] Quesne C 1993 *Phys. Lett.* B **298** 344
Quesne C 1994 *Phys. Lett.* B **322** 344
[33] Fiore G 1998 *J. Phys. A: Math. Gen.* **31** 5289
[34] Quesne C 1999 *Int. J. Theor. Phys.* **38** 1905
[35] Kulish P P, Lyakhovsky V D and Mudrov A I 1998 Extended Jordanian twists for Lie algebras *Preprint* math.QA/9806014